ARTICLES

Stochastic resonance in one-dimensional diffusion with one reflecting and one absorbing end point

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An analysis of the nonmonotonic dependence of the mean-free-passage time on the frequency of a periodic signal [stochastic resonance (SR)] for diffusion on a segment with one absorbing and one reflecting end point shows that SR exists only for some restricted values of parameters. SR always exists if the periodic telegraph signal is replaced by a random one. The latter case is considered in detail.

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I. INTRODUCTION

The stochastic resonance (SR) phenomenon found initially in dynamic nonlinear systems driven by a combination of a random and a periodic field force [1] appears, at first glance, as contradictory. Indeed, resonance, which is a phenomenon usually associated with deterministic systems, arises in stochastic systems as well. In some sense, SR is opposite to the phenomenon of deterministic chaos, studied intensively in the 1960s and 1970s, where the seemingly stochastic features appear in nonlinear dynamic systems. Both phenomena of deterministic chaos and SR show that deterministic and random phenomena complement rather than contradict each other. This discovery, like that of the uncertainly principle, is beyond the scope of nonlinear mechanics, having more general meaning, and represents one of the main achievements of 20th century physics. It is not surprising, therefore, that—again like in the case of deterministic chaos—the number of researchers involved in the study of SR is increasing [2].

We understand the phenomenon of SR in a broad sense, namely, as the nonmonotonic behavior of an output signal or some function of it as a function of some characteristics of the noise or of the periodic signal. Such a widespread definition includes not only SR, *per se*, but also the related phenomena of ''resonance activation,'' ''coherent stochastic resonance,'' etc. However, even with the large variety of models of SR that have been analyzed and different applications, there is still incomplete understanding of the conditions that may be necessary to produce SR. At first, the impression was that all three ingredients, nonlinearity, periodic, and random forces, are necessary for the appearance of SR. Then, it became clear that SR is generated not only in a typical two-well but also in a monostable potential both in the overdamped [3] and in the underdamped [4] cases. Moreover, it turns out that SR occurs even when each one of these ingredients is absent. Indeed, SR exists in linear systems when the additive noise is replaced by nonwhite multiplicative noise [5] or, in slightly artificial systems without internal dynamics at all [6]. The periodic signal cannot be replaced by a constant force in overdamped systems [7] but with such a replacement SR remains in underdamped systems [8]. In turn, deterministic chaos may induce the onset of SR instead of a random signal [9].

Slightly different are the systems where a particle is diffusing on a line terminated by one or two traps. Such systems may be said to be characterized by two states, one in which the particle is untrapped and the second in which it is trapped. These two states appear instead of two states in a double-well potential. Quite surprisingly, it turned out as early as in 1988 [10] that the mean first passage time (MFPT), until the particle is trapped by one of the end points, behaves nonmonotonically as a function of both the frequency [10] and the amplitude of the periodic force [11]. Similar phenomena exist also in a one-dimensional diffusion process on a semi-infinite line terminated by a trapping point driven by a periodic force and a constant bias (to ensure that the MFPT will be finite) [12]. According to our terminology these phenomena also come under the heading of SR.

As in the previously mentioned examples, the characteristic frequency that one needs for a ''resonance'' can be supplied to a system not only by the external periodic force but also by the inverse correlation time of the nonwhite noise, which can be applied to a system in addition to thermal white noise. A dichotomous noise (random telegraph signal) is the simplest form of nonwhite noise. The MFPT for a diffusive particle subjected to dichotomous noise has been calculated by constructing of all possible trajectories in [13,14], and for both white and dichotomous noises in [15]. However, the topic of SR was not raised in these articles of the 1980s, although it was inherent therein.

When the phenomenon of SR is being discussed in the 1990s for these linear systems, it turns out that the boundary conditions play a crucial role, namely SR (which was called in this case ''the coherent stochastic resonance'') exists only when one of the boundaries is absorbing and the other reflecting [16], while it is absent when both boundaries are absorbing [17]. The same crucial influence of the boundary condition appears in a different problem of jumps in a linear double-well potential when the slope of the potential randomly fluctuates between two values at a rate γ. The nonmonotonic dependence of the MFPT on γ (called ''the resonance activation'') [18]—see also [19]—again exists only when the boundary conditions are different at two endpoints. Actually the similarity between these two problems—one-dimensional diffusion under the influence of white and dichotomous noise, and the jumps through fluctuating linear potential—is not so surprising since they are described by the same differential equations.

The aim of this article is the analytical study of SR for one-dimensional diffusion on the segment under the influence of a telegraph signal for the case when one end point of this segment is absorbing while the other is reflecting. Two cases will be considered: one of the periodic telegraph signal, and the other of a random telegraph signal (dichotomous noise).

Since the calculation techniques are quite different for these two cases, we will consider them separately.

We explore further the previously indicated [20,21] method of finding the sufficient (not necessary) condition for detecting SR in complicated problems where the exact solution is not available. If one wants to find whether the dependence of a characteristic time (MFPT in our case) on the frequency of an external field, or on the noise correlation time, or on the rate of fluctuations of some internal parameter is non-monotonic, the following method is suggested. First, one has to find—by some approximate method—the two limit values of the characteristic time for zero and infinite values of the argument. In order to know whether SR (non-monotonic dependence) exists one has to find the asymptotic dependences for very small and very large arguments. If the characteristic time approaches the largest (smallest) limit value from above (below), the nonmonotonic behavior will be the geometric consequence. We illustrate this procedure in the case of periodic signal, and check it in the case of random signal where the exact analytical solution is available.

II. RANDOM TELEGRAPH SIGNAL

The state variable $x(t)$ is described in the overdamped regime by the following Langevin equation

$$\frac{dx}{dt}=\xi(t)+f(t), \tag{1}$$

where $\xi(t)$ is zero-mean uncorrelated white noise whose second-order moment is $\langle \xi(t)\xi(t_1)\rangle = 2D\,\delta(t-t_1)$, in which D is the diffusion constant. The second component of the noise, $f(t)$, is the symmetric dichotomous noise (random telegraph signal) that is allowed to take of the values v and -v. The jump rates for the transition between these two states will be denoted by 2τ so that τ^{-1} is the noise correlation time.

The calculation of the MFPT, $T(x_0)$, for the time to trap a particle, initially at x_0, diffusing on a line segment $(0,L)$ terminated by a trap at $x=0$ and by a reflecting point at $x=L$, is a well-known procedure [22]. One starts with the Fokker-Plank equation for the probability density $p(x,t)$ associated with the Langevin equation (1). Integration of this equation over time and space gives the differential equation for the mean first passage time $T=\int_0^L dx \int_0^\infty dt\, p(x,t)$. The appropriate equation [16]

$$D^2\frac{d^4T}{d{x_0}^4}-(v^2+D\tau)\frac{d^2T}{d{x_0}^2}=\tau \tag{2}$$

has to be solved subject to the boundary conditions at the reflecting boundary $x_0=L$

$$\frac{dT}{dx_0}=0 \quad \text{and} \quad \frac{d^2T}{d{x_0}^2}=-\frac{1}{D}, \tag{3}$$

and at the absorbing boundary $x_0=0$,

$$T=0 \quad \text{and} \quad D^2\frac{d^3T}{d{x_0}^3}-v^2\frac{dT}{dx_0}=0. \tag{4}$$

Skipping a considerable amount of routine algebra we obtain the solution of Eq. (2) subjected to the boundary conditions (3) and (4) as

$$\begin{aligned}T=&-\frac{x_0^2\tau}{2(v^2+D\tau)}+\frac{v^2}{\mu^3[v^2+D\tau\cosh(\mu L)]}\\&\times\Bigg\{\frac{L\tau}{D^2}\sinh[\mu(x_0-L)]-\frac{v^2}{\mu D^3}\cosh[\mu(L-x_0)]\\&-\frac{\tau}{\mu D^2}\cosh(\mu x_0)+\frac{\tau}{\mu D^2}+\frac{v^2}{\mu D^3}\cosh(\mu L)\\&+\frac{L\tau}{D^2}\sinh(\mu L)+\frac{\mu\tau D x_0}{v^2}\Bigg[\frac{L\tau}{D^2}\cosh(\mu L)\\&+\frac{v^2}{\mu D^3}\sinh(\mu L)\Bigg]\Bigg\},\end{aligned} \tag{5}$$

where $\mu=(v^2/D^2+\tau/D)^{1/2}$.

There are three characteristic times in our problem. The first one $\tau_1=L^2/D$, has its origin in the diffusion process, the second one $\tau_2=\tau^{-1}$ is the correlation time of the noise, and the third one $\tau_3=L/v$, represents the strength of noise for the given problem. It is convenient, therefore, to introduce two dimensionless parameters $\alpha=\tau_3/\tau_1=D/vL$ and $\beta=\tau_3/\tau_2=L\tau/v$. Using these parameters and introducing the dimensionless length $y=x_0/L$ one can rewrite Eq. (5) in the following form:

$$\begin{aligned}T=&\frac{L^2}{2D}\Bigg\{-\frac{\alpha\beta y^2}{1+\alpha\beta}\\&+\frac{1}{\nu^3[1+\alpha\beta\cosh(\nu)]}\Bigg[\frac{4\beta}{\alpha}\sinh\left(\frac{\nu y}{2}\right)\cosh\left(\frac{\nu(2-y)}{2}\right)\\&+\frac{4}{\alpha^2\nu}\sinh\left(\frac{\nu y}{2}\right)\sinh\left(\frac{\nu(2-y)}{2}\right)-\frac{4\beta}{\alpha\nu}\sinh^2\left(\frac{\nu y}{2}\right)\\&+y\frac{2\beta}{\alpha}\sinh(\nu y)+2_y\nu\beta^2\cosh(\nu y)\Bigg]\Bigg\},\end{aligned} \tag{6}$$

where $\nu=\mu L=(1/\alpha^2+\beta/\alpha)^{1/2}$. A few limiting cases are of interest. When $\alpha\gg 1$ for fixed β, or $\beta\gg 1$ for fixed α, Eq. (6) reduces to the well-known expression [22] for the MFPT for diffusion on a segment with one absorbing $(x=0)$ and one reflecting $(x=L)$ end points,

$$T=-\frac{x_0^2}{2D}+\frac{x_0 L}{D}. \tag{7}$$

This result is quite natural since $\alpha\gg 1$ or $\tau_1\gg\tau_3$ means the diffusion is the fastest and, therefore, the decisive factor, while $\beta\gg 1$ or $\tau_2\ll\tau_3$ implies that for very high rate, the dichotomous noise does not influence the MFPT.

For low rate of dichotomous noise, $\beta\ll 1$, Eq. (6) reduces to

$$T=\frac{2D}{v^2}\sinh\left(\frac{v x_0}{2D}\right)\sinh\left(\frac{v(2L-x_0)}{2D}\right). \tag{8}$$

A comparison between Eqs. (7) and (8) shows that the limit expression (8) for small rate of dichotomous noise is always larger than the limit expression (7) for large rates, and reduces to it only if, in addition, $\alpha\gg 1$. The latter result could be expected since the conditions $\alpha\gg 1$ and $\beta\ll 1$ mean $\tau_2>\tau_3>\tau_1$, i.e., the diffusion process is the fastest and then the decisive one.

Hence, we have obtained that the MFPT for zero rate is always larger than that for infinite rates of the dichotomous noise. If we can show now that the MFPT is increasing with τ for large rates approaching the limit value (7), then the existence of a minimum of the MFPT as a function of τ will be a geometrical consequence.

To simplify the further analysis let us choose the initial position of a particle at the reflective boundary, $x_0=L$ or $y=1$. Then Eq. (6) takes the following form

$$T=\frac{L^2}{2D}\left\{-\frac{\alpha\beta}{1+\alpha\beta}+\frac{1}{\nu^3[1+\alpha\beta\cosh(\nu)]}\left[\frac{4\beta}{\alpha}\sinh(\nu)-\frac{4\beta}{\alpha\nu}\sinh^2\left(\frac{\nu}{2}\right)+2\nu\beta^2\cosh(\nu)+\frac{4}{\nu\alpha^2}\sinh^2\left(\frac{\nu}{2}\right)\right]\right\} \tag{9}$$

and its expansion for large $\beta=L\tau/v$ is

$$T=\frac{L^2}{2D}\left(1-\frac{1}{\alpha\beta}+\cdots\right), \tag{10}$$

i.e., the MFPT approaches the limit value at infinite noise rate (smaller than that at zero rate) from below, which is evidence for the existence of SR.

At the same time, the asymptotic value of Eq. (9) for small β has the following form

$$T=\frac{L^2}{2D}\left\{4\alpha^2\sinh^2\left(\frac{1}{2\alpha}\right)-\beta\left[\alpha+4\alpha^2\sinh^2\left(\frac{1}{\alpha}\right)+4\alpha^3\sinh^2\left(\frac{1}{2\alpha}\right)\right]+\cdots\right\}, \tag{11}$$

which shows that the MFPT with the increasing noise rate $\beta=L\tau/v$ decreases from its (larger than that at infinite rate) limit value at $\beta=0$, i.e., no conclusions concerning the existence of an additional maximum of the MFPT can be made from this analysis.

III. PERIODIC TELEGRAPH SIGNAL

We return now to the original equation (1) where, however, the random telegraph signal $f(t)$ is replaced by the periodic one

$$f(t)=\begin{cases}+v & \text{for } t\epsilon[2n\Gamma,(2n+1)\Gamma]\\ -v & \text{for } t\epsilon[(2n+1)\Gamma,(2n+2)\Gamma],\end{cases} \tag{12}$$

where Γ is the period of the telegraph signal, and $n=0,1,2,\ldots$. The rate (frequency) ω of the signal is equal to $\omega=(2\Gamma)^{-1}$.

The Fokker-Planck equation corresponding to the Langevin equation (1)

$$\frac{\partial p_\pm}{\partial t}=\pm v\frac{\partial p_\pm}{\partial x}+D\frac{\partial^2 p_\pm}{\partial x^2}, \tag{13}$$

subjected to the initial conditions $p_\pm(x,t=0|x_0)=\delta(x-x_0)$, has a well-known [23] solution on the segment $(0,L)$ terminated by the reflecting point $x=0$ and by absorbing point $x=L$

$$p_\pm(x,t|x_0)=\frac{2}{L}\exp\left[\frac{\pm v(x-x_0)}{2D}-\frac{v^2 t}{4D}\right]\times\sum_{n=0}^{\infty}\cos(\beta_n x)\cos(\beta_n x_0)\exp(-D\beta_n^2 t), \tag{14}$$

where $\beta_n=[(2n+1)\pi]/2L$. The general solution of our problem can be obtained now by matching the solutions (14) corresponded for bias $\pm v$ at the times at which the telegraph signal (12) changes sign. If one defines $P_k(x), k=0,1,\ldots,$ as the probabilities that at time $t=k\Gamma$ when the bias changes its sign the system is found at position $x,x+dx$, the quantities $P_k(x)$ can be found [20] from the obvious recurrence relations

$$P_{m+1}(x)=\int_0^L p_\pm(x,\Gamma|y)P_m(y)dy, \tag{15}$$

where the signs "+" and "−" in Eq. (15) correspond to $m=2n$ and $m=2n+1$, respectively, and the initial condition $P_0(y)=\delta(y-x_0)$, which leads to

$$P_1(y)=p_+(y,\Gamma|x_0). \tag{16}$$

Then, the mean-free-passage time T turns out to be equal [24,20]

$$T=\sum_{n=0}^{\infty}\int_0^\Gamma dt\int_0^L dx\int_0^L dy[p_+(x,t|y)P_{2n}(y)+p_-(x,t|y)P_{2n+1}(y)]. \tag{17}$$

The limit expressions for infinite and zero frequencies can be found independently using the differential equations for the MFPT [22]. For $\omega \to \infty$, the quickly oscillating signal, like in the case of dichotomous noise, does not influence the MFPT, which is defined only by pure diffusion, and turns out to be equal [22]

$$T(\omega \to \infty) = \frac{L^2 - x_0^2}{2D}. \tag{18}$$

Notice that Eq. (18) is slightly different from Eq. (7) since now $x=0$ is the reflecting point, and $x=L$ is the absorbing one.

For the opposite limit case $\omega=0$, the MFPT for a system driven by white noise and a constant bias can be easily found [22]:

$$T(\omega=0) = \frac{L-x_0}{v} + \frac{D}{v^2}\left[\exp\left(-\frac{vL}{D}\right) - \exp\left(-\frac{vx_0}{D}\right)\right]. \tag{19}$$

The latter equation can be also obtained by substituting (14) and (16) into Eq. (17) with $\Gamma=\infty$, i.e.,

$$\begin{aligned} T(\omega=0) &= \int_0^\infty dt \int_0^L dx\, p_+(x,t|x_0) \\ &= \frac{2D}{L}\exp\left(-\frac{vx_0}{2D}\right) \\ &\quad \times \sum_{n=0}^{\infty} \frac{\left[(-1)^n \beta_n \exp\left(\frac{vL}{2D}\right) - \frac{v}{2D}\right]}{\left(D\beta_n^2 + \frac{v^2}{4D}\right)^2} \\ &\quad \times \cos(\beta_n x_0). \end{aligned} \tag{20}$$

A comparison between Eqs. (18) and (19) shows that, as opposed to the case of dichotomous noise, for the periodic telegraph signal $T(\omega=0) \ll T(\omega \to \infty)$. In order to find the small corrections in ω to Eq. (19) one has to find the appropriate terms in the general expression (17) for the MFPT. We use here the straitforward procedure rather than the sophisticated methods used in [24,20].

According to Eq. (14), the time-dependent terms in Eq. (17) are of the form $\exp[-(D\beta_n^2 + v^2/4D)t]$, i.e., after integration over t they will transform to $\exp[-(D\beta_n^2 + v^2/4D)/2\omega]$. This strong exponential dependence on ω allowes us to leave only the $n=0$ terms in Eq. (17) which, after using Eq. (16), leads to the following expression for the the MFPT for small ω

$$\begin{aligned} T(\omega) \approx & \int_0^\Gamma dt \int_0^L dx\, p_+(x,t|x_0) \\ & + \int_0^\Gamma dt \int_0^L dx \int_0^L dy\, p_-(x,t|y) p_+(y,\Gamma|x_0). \end{aligned} \tag{21}$$

Retaining only the leading term of the MFPT expansion around $\omega=0$, which is of order $\exp[-D\beta_0^2 + v^2/4D/2\omega]$, it is possible to keep only $n=0$ terms in the p_+ functions in Eq. (21), although the sum entering p_- function has to be retained. Under these assumptions, using Eq. (14) and performing integration over x in the second term in (21) one can rewrite the leading term in the latter equation in the following form:

$$\begin{aligned} T(\omega) = & \frac{2}{L}\cos(\beta_0 x_0)\exp\left(-\frac{vx_0}{2D}\right)\exp\left[-\frac{D\beta_0^2 + \frac{v^2}{4D}}{2\omega}\right] \\ & \times \left\{ -\frac{\int_0^L \cos(\beta_0 x)\exp\left(\frac{vx}{2D}\right)dx}{\left(D\beta_0^2 + \frac{v^2}{4D}\right)} \right. \\ & + \int_0^L dy \cos(\beta_0 y)\exp\left(\frac{vy}{2D}\right) \\ & \times \sum_{n=0}^{\infty} \frac{2D}{L} \frac{\left[(-1)^n \beta_n \exp\left(-\frac{vL}{2D}\right) + \frac{v}{2D}\right]}{\left(D\beta_n^2 + \frac{v^2}{4D}\right)^2} \\ & \left. \times \cos(\beta_n y)\exp\left(\frac{vy}{2D}\right) \right\}. \end{aligned} \tag{22}$$

There is no need to calculate the sum in Eq. (22) since the same sum—with opposite sign of v—already appeared in Eq. (20), and one can use the result of summation given by Eq. (19). Substituting (19) with the opposite sign of v into (22) one gets

$$\begin{aligned} T(\omega) = & \frac{2}{L}\cos(\beta_0 x_0)\exp\left(-\frac{vx_0}{2D}\right)\exp\left[-\frac{D\beta_0^2 + \frac{v^2}{4D}}{2\omega}\right] \\ & \times \left\{ -\frac{\int_0^L \cos(\beta_0 x)\exp\left(\frac{vx}{2D}\right)dx}{\left(D\beta_0^2 + \frac{v^2}{4D}\right)} \right. \\ & + \int_0^L dy \cos(\beta_0 y)\exp\left(\frac{vy}{2D}\right) \\ & \left. \times \left\{ -\frac{L-y}{v} + \frac{D}{v^2}\left[\exp\left(\frac{vL}{D}\right) - \exp\left(\frac{vy}{D}\right)\right]\right\} \right\}. \end{aligned} \tag{23}$$

Performing integration in Eq. (23) one obtains the expansion of the MFPT near $\omega=0$,

$$T\approx\frac{L^2}{2D}\exp\left(-\frac{\gamma x_0}{2L}\right)\cos\left(\frac{\pi x_0}{2L}\right)\exp\left[-(\gamma^2+\pi^2)\frac{D}{8L^2\omega}\right]$$
$$\times\left\{\frac{16}{(\pi^2+\gamma^2)^2}\left[-4\pi\exp\left(\frac{\gamma}{2}\right)+3\gamma-\frac{\pi^2}{\gamma}\right]\right.$$
$$+\frac{8}{\gamma(\pi^2+\gamma^2)}\left[\frac{\pi}{\gamma}\exp\left(\frac{3\gamma}{2}\right)-\exp(\gamma)+\gamma\right]$$
$$\left.-\frac{8}{\gamma(\pi^2+9\gamma^2)}\left[\frac{\pi}{\gamma}\exp\left(\frac{3\gamma}{2}\right)-3\right]\right\}, \tag{24}$$

where $\gamma=vL/D$. Analysis of Eq. (24) shows that this expression is positive for $\gamma>\gamma_0=10^{-6}$, and negative for $\gamma<\gamma_0$, i.e., for small $\gamma<\gamma_0$ the MFPT decreases with ω starting from its minimal value (19) at $\omega=0$. Therefore, SR exists in this case. However, for larger $\gamma>\gamma_0$ one cannot draw a definite conclusion as to whether SR exists for such γ from this asymptotic analysis that defines only the sufficient conditions for the existence of SR, and numerical simulation is necessary.

IV. CONCLUSIONS

The overdamped motion of a particle subjected to dichotomous noise has a long history. As we show in the Appendix, SR already appeared in [13,14], although no special attention had been focused on this subject in the middle 1980s. Without going into the long history we should point out two recent articles. We have obtained [16] the general solution of equation of motion of a particle subjected to white Gaussian and asymmetric dichotomous noise. However, the results are presented in quite a complicated form suitable only for numerical analysis. The authors of [19] considered the problem mathematically equivalent to ours, but their analysis is directed towards approximate detecting of the location of the minimum and the minimal value of the MFPT. Our analysis of the exact result and all limit cases complement the studies performed in [16,19]. Analyzing the limit behavior of the MFPT we have shown the efficiency of the geometrical method of detecting an existence of SR. This method becomes of crucial importance in the cases when the exact solution is not available, as we show in the example of a motion under the influence of the periodic telegraph signal.

The latter problem has been considered for the diffusion on a line with two absorbing end points [24,20], where quite complicated analyses of the low-frequency behavior have been used. We suggested a much simpler method for such analysis. The results obtained for our case of one absorbing and one reflecting end point are quite remarkable, namely, it turns out that SR exists for some small value of the dimensionless parameter $\gamma=vL/D$, while for a larger value of this parameter our (sufficient) method shows no SR that, of course, cannot be considered as a proof of its absence. Notice that an existence of SR only for restricted values of parameters appears in some other cases as well (see, for example, the Appendix or [25]). Since no SR appears in the case of two absorbing boudaries [20] one concludes that the asymmetric boundary conditions favor an onset of SR for a periodic telegraph signal.

The intriguing problem of the necessary and sufficient conditions for the onset of SR still remains open.

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APPENDIX

The MFPT for a particle located initially at point x_0, diffusing on a segment $(-L,L)$ with two absorbing boundaries and subjected to asymmetric dichotomous noise has the following form [14]

$$T=\frac{\lambda_a(a+b)(L^2-x_0^2)}{2a^2b}+\frac{L-x_0}{a}\,\frac{\dfrac{a}{\lambda_a}+\dfrac{L(a+b)}{b}}{\dfrac{a}{\lambda_a}+2L}. \tag{A1}$$

One can easily see from Eq. (A1) that the MFPT has an extremum, i.e., the equation $dT/d\lambda_a=0$ has solutions if $a>b$. Hence,we have proved the nonmonotonic dependence of T as function of λ_a (SR) for this simple case.

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